Exponential equations

Since the exponential function is"1-1" and "on" mapping we can use: $a^{f(x)} = a^{g(x)} \Leftrightarrow f(x) = g(x)$

This means that when both sides make the same basis, we compared exponents.

Here are a few examples:

1) Solve the equations:

a) $4^x = 2^{\frac{x+1}{x}}$
b) $8^{x+1} = 16 \cdot 2^{x-2}$
v) $16^{\frac{1}{x}} = 4^{\frac{x}{2}}$
g) $16 \cdot 2^{5x+2} = 2^{x^2}$
d) $9^{-3x} = \left(\frac{1}{27}\right)^{x+3}$
m) $\left(x^2+1\right)^{2x-3} = 1$
e) $9^{x^2-3x+5} = 3^6$

Solutions:

a)

$$4^x = 2^{\frac{x+1}{x}}$$

$$(2^2)^x = 2^{\frac{x+1}{x}}$$

$$2^{2x} = 2^{\frac{x+1}{x}} \quad \Leftrightarrow \quad 2x = \frac{x+1}{x}$$

$$2x^2 = x+1$$

$$2x^2 - x - 1 = 0$$

$$x_{1,2} = \frac{1 \pm 3}{4}$$

$$x_1 = 1$$

$$x_2 = -\frac{1}{2}$$

Solutions are: $x_1 = 1$ and $x_2 = -\frac{1}{2}$

b) $8^{x+1}=16\cdot 2^{x-2}$

$(2^3)^{x+1}=2^4\cdot 2^{x-2}$

$2^{3x+3}=2^{4+x-2}$

$2^{3x+3}=2^{x+2}$

$3x+3=x+2$

$3x-x=2-3$

$2x=-1$

$x=-\frac{1}{2}$

v) $16^{\frac{1}{x}}=4^{\frac{x}{2}}$

$(2^4)^{\frac{1}{x}}=(2^2)^{\frac{x}{2}}$

$2^{4\cdot\frac{1}{x}}=2^{2\cdot\frac{x}{2}}$

$2^{\frac{4}{x}}=2^x$

$\frac{4}{x}=x$

$x^2=4$

$x=\pm\sqrt{4}$

$x_1=2$

$x_2=-2$

g) $16\cdot 2^{5x+2}=2^{x^2}$

$2^4\cdot 2^{5x+2}=2^{x^2}$

$2^{4+5x+2}=2^{x^2}$

$2^{5x+6}=2^{x^2}$

$x^2=5x+6$

$x^2-5x+6=0$

$x_{1,2}=\frac{5\pm 1}{2}$

$x_1=3$

$x_2=2$

d) $9^{-3x}=\left(\frac{1}{27}\right)^{x+3}$

$(3^2)^{-3x}=(3^{-3})^{x+3}$

$3^{-6x}=3^{-3x-9}$

Take heed:

$\frac{1}{27}=\frac{1}{3^3}=3^{-3}$

$-6x=-3x-9$

$-6x+3x=-9$

$-3x=-9$

$x=3$

m) $(x^2+1)^{2x-3}=1$

Since we know that $a^o=1$, one solution will give us

$2x-3=0$

$2x=3$

$x=\frac{3}{2}$

Another solution would be if: $x^2+1=1\Rightarrow x^2=0\Rightarrow x=0$

e) $9^{x^2-3x+5}=3^6$

$(3^2)^{x^2-3x+5}=3^6$

$3^{2x^2-6x+10}=3^6$

$2x^2-6x+10=6$

$2x^2-6x+4=0/:2$

$x^2-3x+2=0$

$x_{1,2}=\frac{3\pm 1}{2}$

$x_1=2$

$x_2=1$

2) Solve the equations:

a) $2^{x+3} - 7 \cdot 2^{x} - 16 = 0$
b) $3^{x-1} - 4 \cdot 3^{x} + 33 = 0$
c) $2 \cdot 3^{x+1} - 4 \cdot 3^{x-2} = 450$
d) $2^{3x-2} - 2^{3x-3} - 2^{3x-4} = 4$
e) $2^{x-1} - 2^{x-3} = 3^{x-2} - 3^{x-3}$

Solutions:

Here we use the rules for degrees:

$$a^{m+n} = a^{m} \cdot a^{n}$$
$$a^{m-n} = \frac{a^{m}}{a^{n}}$$
$$\left(a^{m}\right)^{n} = a^{m \cdot n}$$

a) $2^{x+3} - 7 \cdot 2^{x} - 16 = 0$

$2^{x} \cdot 2^{3} - 7 \cdot 2^{x} - 16 = 0 \rightarrow$ replacement $2^{x} = t$

$t \cdot 8 - 7 \cdot t - 16 = 0$

$8t - 7t = 16$

$t = 16 \rightarrow$ Go back to replacement $2^{x} = t$

$2^{x} = 16$

$2^{x} = 2^{4}$

$x = 4$

b) $3^{x-1} - 4 \cdot 3^{x} + 33 = 0$

$\frac{3^{x}}{3} - 4 \cdot 3^{x} + 33 = 0 \rightarrow$ replacement $3^{x} = t$

$\frac{t}{3} - 4t + 33 = 0 \rightarrow$ multiply all with 3

$t - 12t + 99 = 0$

$-11t = -99$

$t = 9$

$3^{x} = 9$

$3^{x} = 3^{2}$

$x = 2$

c) $2\cdot 3^{x+1}-4\cdot 3^{x-2}=450$

$2\cdot 3^{x}\cdot 3^{1}-4\frac{3^{x}}{3^{2}}=450\rightarrow$ replacement $3^{x}=t$

$6\cdot t-4\frac{t}{9}=450$

$6t-\frac{4t}{9}=450\rightarrow$ multiply all with 9

$54t-4t=4050$

$50t=4050$

$t=\frac{4050}{50}$

$t=81$

$3^{x}=81\rightarrow 81=3\cdot 3\cdot 3\cdot 3=3^{4}$

$3^{x}=3^{4}$

$x=4$

d) $2^{3x-2}-2^{3x-3}-2^{3x-4}=4$

$\frac{2^{3x}}{2^{2}}-\frac{2^{3x}}{2^{3}}-\frac{2^{3x}}{2^{4}}=16\rightarrow$ replacement $2^{3x}=t$

$\frac{t}{4}-\frac{t}{8}-\frac{t}{16}=16\rightarrow$ multiply all with 16

$4t-2t-t=256$

$t=256$

$2^{3x}=2^{8}$

$3x=8$

$x=\frac{8}{3}$

e) $2^{x-1}-2^{x-3}=3^{x-2}-3^{x-3}$

$\frac{2^{x}}{2}-\frac{2^{x}}{2^{3}}=\frac{3^{x}}{3^{2}}-\frac{3^{x}}{3^{3}}$

$\frac{2^{x}}{2}-\frac{2^{x}}{8}=\frac{3^{x}}{9}-\frac{3^{x}}{27}$

$\frac{4\cdot 2^{x}-2^{x}}{8}=\frac{3\cdot 3^{x}-3^{x}}{27}$

$\frac{3\cdot 2^{x}}{8}=\frac{2\cdot 3^{x}}{27}$

$3\cdot 2^{x}\cdot 27=2\cdot 3^{x}\cdot 8 \longrightarrow 2^{x}\cdot 81=3^{x}\cdot 16$

$\frac{2^{x}}{3^{x}}=\frac{16}{81}$

$\left(\frac{2}{3}\right)^{x}=\left(\frac{2}{3}\right)^{4}$

$x=4$

3) Solve the equations:

a) $4^x - 5 \cdot 2^x + 4 = 0$
b) $16^x - 4^x - 2 = 0$
c) $5^x - 5^{3-x} = 20$
d) $5^{2x-3} = 2 \cdot 5^{x-2} + 3$
e) $(11^x - 11)^2 = 11^x + 99$

Solutions:

a) $4^x - 5 \cdot 2^x + 4 = 0 \rightarrow$ Because $4^x = (2^2)^x = 2^{2x}$ replacement is $2^x = t \longrightarrow 4^x = t^2$

$t^2 - 5t + 4 = 0$

$t_{1,2} = \frac{5 \pm 3}{2}$

$t_1 = 4$

$t_2 = 1$

$2^x = 4$

$2^x = 2^2$ or $2^x = 1$

$x = 2$ $x = 0$

b) $16^x - 4^x - 2 = 0 \rightarrow$ replacement is $4^x = t$, and then $16^x = 4^{2x} = t^2$

$t^2 - t - 2 = 0$

$t_{1,2} = \frac{1 \pm 3}{2}$

$t_1 = 2$

$t_2 = -1$

$4^x = 2$

$2^{2x} = 2^1$

$2x = 1$ or $4^x = -1 \longrightarrow$ here there is no solution, because $y = a^x$ is always positive!

$x = \frac{1}{2}$

c) $5^x - 5^{3-x} = 20$

$5^x - \frac{5^3}{5^x} = 20 \rightarrow$ replacement $5^x = t$

$t - \frac{125}{t} = 20 \rightarrow$ multiply all with t

$t^2 - 125 = 20t$

$t^2 - 20t - 125 = 0$

$t_{1,2} = \frac{20 \pm 30}{2}$

$t_1 = 25$

$t_2 = -5$

$5^x = 25$ or $5^x = -5$ $\longrightarrow$ No solutions

$5^x = 5^2$

$x = 2$

d) $5^{2x-3} = 2 \cdot 5^{x-2} + 3$

$\frac{5^{2x}}{5^3} = 2 \cdot \frac{5^x}{5^2} + 3 \rightarrow$ replacement $5^x = t$

$\frac{t^2}{125} = \frac{2t}{25} + 3 \rightarrow$ multiply all with 125

$t^2 = 10t + 375$

$t^2 - 10t - 375 = 0$

$t_{1,2} = \frac{10 \pm 40}{2}$

$t_1 = 25$

$t_2 = -15$

Back in the replacement:

$5^x = 25$

$5^x = 5^2$ or $5^x = -15$ no solution, because $5^x > 0$

$x = 2$

e) $(11^x - 11)^2 = 11^x + 99 \rightarrow$ replacement $11^x = t$

$(t-11)^2 = t + 99$

$t^2 - 22t + 121 - t - 99 = 0$

$t^2 - 23t + 22 = 0$

$t_{1,2} = \frac{23 \pm 21}{2}$

$t_1 = 22$

$t_2 = 1$

Back in the replacement:

$11^x = 22$ or $11^x = 1$

$x = \log_{11} 22$ $x = 0$

4. **Solve the equations:**

a) $4^{\sqrt{x-2}}+16=10\cdot 2^{\sqrt{x-2}}$

b) $4^{x+\sqrt{x^2-2}}-5*2^{x-1+\sqrt{x^2-2}}=6$

c) $\left(\sqrt{2+\sqrt{3}}\right)^x+\left(\sqrt{2-\sqrt{3}}\right)^x=4$

Solutions:

a) $4^{\sqrt{x-2}}+16=10\cdot 2^{\sqrt{x-2}}$

First, determine the area definition : $x-2\geq 0\Rightarrow x\geq 2$

Replacement: $2^{\sqrt{x-2}}=t\Rightarrow 4^{\sqrt{x-2}}=t^2$

$$t^2+16=10t$$
$$t^2-10t+16=0$$
$$t_{1,2}=\frac{10\pm 6}{2}$$
$$t_1=8$$
$$t_2=2$$

$$2^{\sqrt{x-2}}=8$$
$$2^{\sqrt{x-2}}=2^3$$
$$\sqrt{x-2}=3$$
$$x-2=9$$
$$x=11$$

or

$$2^{\sqrt{x-2}}=2$$
$$\sqrt{x-2}=1$$
$$x-2=1$$
$$x=3$$

b)

$$4^{x+\sqrt{x^2-2}}-5\cdot 2^{x-1+\sqrt{x^2-2}}=6$$
$$(2^2)^{x+\sqrt{x^2-2}}-5\cdot 2^{x+\sqrt{x^2-2}-1}=6$$
$$2^{2x+\sqrt{x^2-2}}-5\cdot\frac{2^{x+\sqrt{x^2-2}}}{2^1}=6$$

Replacement: $x+\sqrt{x^2-2}=t$

$t^2-\frac{5t}{2}=6$ multiply with 2

$$2t^2 - 5t - 12 = 0$$

$$t_{1,2} = \frac{5 \pm \sqrt{121}}{4} = \frac{5 \pm 11}{4}$$

$$t_1 = 4$$

$$t_2 = -\frac{6}{4} = -\frac{3}{2}$$

$$2^{x+\sqrt{x^2-2}} = 4$$

$$2^{x+\sqrt{x^2-2}} = 2^2$$

$$x + \sqrt{x^2 - 2} = 2$$

$\sqrt{x^2 - 2} = 2 - x \rightarrow$ conditions: $2 - x \geq 0$ and $x^2 - 2 \geq 0$ $\quad -x \geq -2 \quad x \leq 2$

$x^2 - 2 = (2 - x)^2 \qquad\qquad x \in (-\infty, -\sqrt{-2}) \cup (\sqrt{-2}, \infty)$

$$x^2 - 2 = 4 - 4x + x^2$$

$$4x = 4 + 2$$

$$4x = 6$$

$$x = \frac{6}{4} = \frac{3}{2} = 1{,}5$$

$x = 1{,}5 \rightarrow$ Satisfies the conditions

c) $\left(\sqrt{2+\sqrt{3}}\right)^x + \left(\sqrt{2-\sqrt{3}}\right)^x = 4$

Let us see first one thing:

$$2 - \sqrt{3} = \frac{2-\sqrt{3}}{1} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2^2 - \sqrt{3}^2}{2+\sqrt{3}} = \frac{4-3}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}}$$

$$\left(\sqrt{2+\sqrt{3}}\right)^x + \frac{1}{\sqrt{2+\sqrt{3}}^x} = 4$$

Replacement: $\sqrt{2+\sqrt{3}}^x = t$

$t + \frac{1}{t} = 4 \rightarrow$ multiply with t

$$t^2 + 1 = 4t$$

$$t^2 - 4t + 1 = 0$$

$$t_{1,2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = \frac{4 \pm \sqrt{3}}{2} = \frac{2(2 \pm \sqrt{3})}{2} = 2 \pm \sqrt{3}$$

$$t_1 = 2 + \sqrt{3}$$

$$t_2 = 2 - \sqrt{3}$$

Back in the replacement:

$\sqrt{2+\sqrt{3}}^{x} = t$, so:

$\sqrt{2+\sqrt{3}}^{x} = 2+\sqrt{3}$ or $\sqrt{2+\sqrt{3}}^{x} = 2-\sqrt{3}$

$\sqrt[m]{a^n} = a^{\frac{n}{m}} \longrightarrow \sqrt[2]{a^x} = a^{\frac{x}{2}}$

$$(2+\sqrt{3})^{\frac{x}{2}} = (2+\sqrt{3})^{1}$$
$$\frac{x}{2} = 1$$
$$x = 2$$

$$(2+\sqrt{3})^{\frac{x}{2}} = \frac{1}{2+\sqrt{3}}$$
$$(2+\sqrt{3})^{\frac{x}{2}} = (2+\sqrt{3})^{-1}$$
$$\frac{x}{2} = -1$$
$$x = -2$$

5. **Solve the equations:**

a) $20^x - 6\cdot 5^x + 10^x = 0$

b) $6\cdot 9^x - 13\cdot 6^x + 6\cdot 4^x = 0$

Solutions:

a) $20^x - 6\cdot 5^x + 10^x = 0 \rightarrow$ we use that: $(a\cdot b)^n = a^n \cdot b^n$

$$(5\cdot 4)^x - 6\cdot 5^x + (5\cdot 2)^x = 0$$
$$5^x \cdot 4^x - 6\cdot 5^x + 5^x \cdot 2^x = 0$$
$$5^x(4^x - 6 + 2^x) = 0$$
$$5^x = 0 \quad \vee \quad 4^x + 2^x - 6 = 0$$
$$t^2 + t - 6 = 0$$
$$t_{1,2} = \frac{-1\pm 5}{2}$$
$$t_1 = 2$$
$$t_2 = -3$$

$2^x = 2$, $x = 1$ $\vee$ $2^x = -3 \longrightarrow$ no solution

Only solution is x = 1

b) $6\cdot 9^x - 13\cdot 6^x + 6\cdot 4^x = 0$

$$6\cdot 3^{2x} - 13\cdot 3^x\cdot 2^x + 6\cdot 2^{2x} = 0$$

$$6\cdot\frac{3^{2x}}{2^{2x}} - 13\cdot\frac{3^x}{2^x} + 6 = 0$$

$$6\cdot\left(\frac{3}{2}\right)^{2x} - 13\cdot\left(\frac{3}{2}\right)^x + 6 = 0$$

Replacement: $\left(\frac{3}{2}\right)^x = t$

$$6t^2 - 13t + 6 = 0$$

$$t_{1,2} = \frac{13\pm 5}{12}$$

$$t_1 = \frac{18}{12} = \frac{3}{2}$$

$$t_2 = \frac{8}{12} = \frac{2}{3}$$

$\left(\frac{3}{2}\right)^x = \frac{3}{2}$ or $\left(\frac{3}{2}\right)^x = \frac{2}{3}$

$x = 1$

$$\left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^{-1}$$

$x = -1$

6) Graphic solve the following equation:

a) $2^x - 5 + \frac{x}{2} = 0$

b) $3^x - \frac{x}{2} - 8 = 0$

Solutions:

a) First, we separate functions, the exponent on the left, and the rest on the right side:

$$2^x = 5 - \frac{x}{2}$$

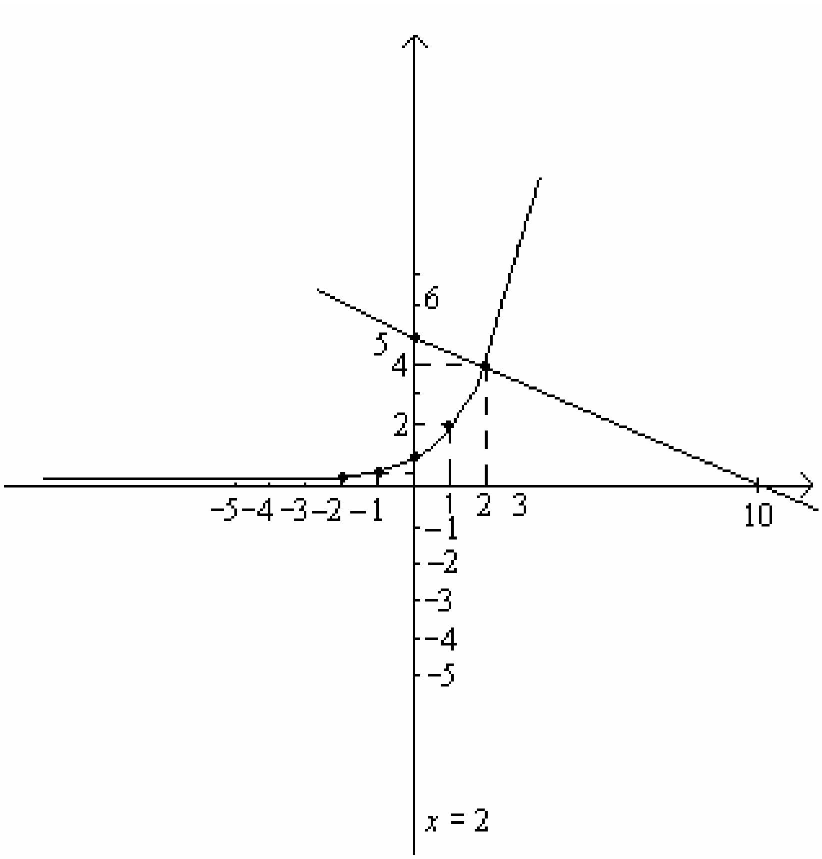
$y = 2^x$ and $y = -\frac{x}{2} + 5$ ⟶ the section will give us a solution.

$$y = 2^x$$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$$y = -\frac{x}{2} + 5$$

X	0	10	2
Y	5	0	4



The solution is x = 2

b) $3^x - \frac{x}{2} - 8 = 0$

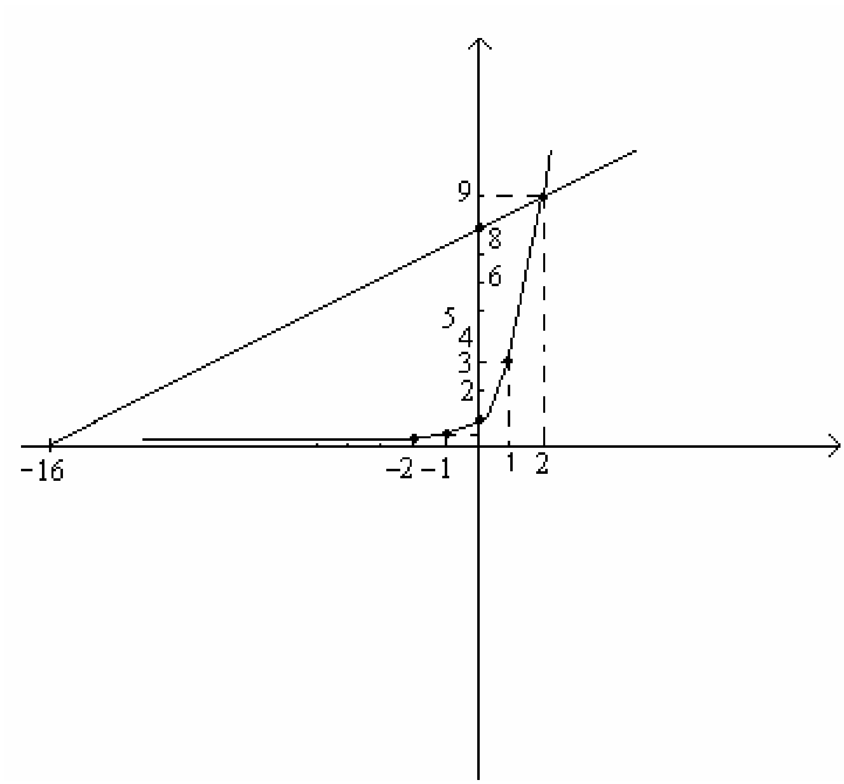
$$3^x = \frac{x}{2} + 8$$

$$y = 3^x$$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

$$y = \frac{x}{2} + 8$$

x	0	10	2
y	8	0	9



So, the solution is x = 2

Is there another solution? Yes, but it can be difficult to find it very precisely.....

We will learn that later...